Brachistochrone Inside the Earth

A brilliant physics student thinks that our transportation problems can be solved by digging a network of tunnels through the Earth connecting different cities (New York and LA, say). The idea is to use the gravitational attraction supplied by the Earth to accelerate a person from one point to another. Let's idealize this and neglect air resistance and any friction which the person may encounter when in contact with the tunnel, and we'll assume that the Earth has a uniform density (and more importantly, we'll neglect the cost of digging the tunnel). The problem is to find the shape of the tunnel which will minimize the transit time between the cities. You should find that the shape is that of a *hypocycloid*, which is the curve described by a point on a circle rolling inside a circle of a larger radius. After finding the shape, find the time required to go from NY to LA (I find about 28 minutes). If you want to solve this problem, you may want to first review Sec. 5.2 of *Marion and Thornton*—you'll need to know the form of the gravitational potential energy *inside* the Earth.

Solution. First, find the gravitational field $\mathbf{g}$ using

$$\oint_S \mathbf{g} \cdot \mathbf{n} \, da = -4\pi G \int_V \rho(\mathbf{x}) \, d^3x. \tag{1}$$

Since the mass density is assumed to be uniform, we have

$$\mathbf{g} = \begin{cases} -\frac{GM}{R^3} r \mathbf{e}_r & r < R \\[2mm] -\frac{GM}{r^2} \mathbf{e}_r & r > R \end{cases} \tag{2}$$

where M is the mass and R is the radius of the Earth. The gravitational force is $\mathbf{F} = m\mathbf{g}$, with m the mass of the particle moving in the tunnel; we need to integrate this once to obtain the potential energy $U(r)$, with the result

$$U(r) = \begin{cases} \frac{GMm}{2R^3} r^2 - \frac{3}{2} \frac{GMm}{R} & r < R \\[2mm] -\frac{GMm}{r} & r > R \end{cases} \tag{3}$$

where we've chosen the potential energy to be zero at $r = \infty$.

The total energy of the particle is

$$E = \frac{1}{2} mv^2 + U(r). \tag{4}$$

Assuming that $v = 0$ at $r = R$ and solving for v, we find

$$v = \frac{ds}{dt} = \sqrt{\frac{g(R^2 - r^2)}{R}}, \tag{5}$$

where $g = GM/R^2 = 9.8$ m/s^2, and ds is an element of arclength. Solving for dt and integrating, we have for the time of descent

$$t = \sqrt{R/g} \int_1^2 \frac{ds}{\sqrt{R^2 - r^2}}. \tag{6}$$

To make the equations somewhat simpler, I'll work with units such that time is measured in units of $\sqrt{R/g}$ and lengths are measured in units of R, in which case our functional becomes

$$t = \int_1^2 \frac{ds}{\sqrt{1-r^2}}. \tag{7}$$

Now we need to choose a coordinate system. I found it easier to work in rectangular coordinates than in polar coordinates. The shortest path will lie in a plane which contains NY, LA, and the center of the Earth, so we have effectively a two-dimensional problem, and we'll use the rectangular coordinates (x, y). Let τ be a parameter which specifies a point on the curve; the arclength is then

$$ds = \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2}\, d\tau. \tag{8}$$

The time is

$$t = \int_1^2 \frac{\sqrt{(x')^2 + (y')^2}}{\sqrt{1 - x^2 - y^2}}, \tag{9}$$

where $x' \equiv dx/d\tau$, $y' \equiv dy/d\tau$. This is a functional of the two functions x and y, so we need to minimize a functional of two functions. The integrand is

$$f(x, x', y, y') = \frac{\sqrt{(x')^2 + (y')^2}}{\sqrt{1 - x^2 - y^2}} \tag{10}$$

which doesn't depend explicitly upon τ. Therefore, we can use the "second" form of Euler's equation,

$$f - x'\frac{\partial f}{\partial x'} = \text{constant}, \quad f - x'\frac{\partial f}{\partial x'} = \text{constant}. \tag{11}$$

Taking the derivatives, we obtain the two coupled equations,

$$\frac{(y')^2}{\sqrt{1 - x^2 - y^2}\sqrt{(x')^2 + (y')^2}} = C_1, \tag{12}$$

$$\frac{(x')^2}{\sqrt{1 - x^2 - y^2}\sqrt{(x')^2 + (y')^2}} = C_2, \tag{13}$$

with C_1 and C_2 constants. If we add these two equations together, we obtain

$$\frac{\sqrt{(x')^2 + (y')^2}}{\sqrt{1 - x^2 - y^2}} = C_1 + C_2 = C, \tag{14}$$

with C yet another constant. Squaring both sides and simplifying,

$$(x')^2 + (y')^2 = C^2(1 - x^2 - y^2). \tag{15}$$

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The solution of this equation is a *hypocycloid* (see the discussion at the end of the problem), described by

$$x(\tau) = (1 - b)\cos\tau + b\cos\left(\frac{1-b}{b}\tau\right),\tag{16}$$

$$y(\tau) = (1 - b)\sin\tau - b\sin\left(\frac{1-b}{b}\tau\right),\tag{17}$$

so that

$$1^2 - x^2 - y^2 = 2b(1 - b)[1 - \cos(\tau/b)],\tag{18}$$

$$(x')^2 + (y')^2 = 2(1 - b)^2[1 - \cos(\tau/b)].\tag{19}$$

Substituting into Eq. (15), we find

$$2(1 - b)^2 = C^2 2b(1 - b),\tag{20}$$

so that the constant C is

$$C = \sqrt{\frac{1-b}{b}}.\tag{21}$$

With our solution, we can find the transit time:

$$t = \int_1^2 C\,d\tau = \sqrt{\frac{1-b}{b}}(\tau_2 - \tau_1).\tag{22}$$

What is $\tau_2 - \tau_1$? To find this calculate r:

$$r^2 = x^2 + y^2 = 1 + 2b(1 - b)[\cos(\tau/b) - 1].\tag{23}$$

The starting and ending points are on the Earth's surface, where $r = 1$, which requires that $\tau/b = 2\pi n$, with n an integer. The *change* in τ between successive excursions to the Earth's surface is then $\Delta\tau = 2\pi b$, and therefore the transit time is

$$t = 2\pi\sqrt{b(1 - b)}.\tag{24}$$

We could also write this in terms of the angle θ_0 subtended by the radius vector connecting the center of the Earth to the two cities: $\theta_0 = 2\pi b$. If s is the distance between cities measured along the Earth, then $\theta_0 = s/R$, so that $b = s/2\pi R$. Our final result is then (returning to conventional units)

$$t = 2\pi\sqrt{\frac{R}{g}}\sqrt{\frac{s}{2\pi R}\left(1 - \frac{s}{2\pi R}\right)}.\tag{25}$$

Note that when $s = \pi R$, so that the two cities are on opposite sides of the Earth, the path is a straight line with a transit time

$$t = \pi\sqrt{\frac{R}{g}},\tag{26}$$

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exactly as we would expect. Note that the maximum depth under the Earth's surface is $2b = s/\pi$.

For the trip from NY to LA, $s/2\pi R \approx 1/8$, so that $t \approx 28$ minutes. The maximum depth of the tunnel is about 1600 km. A straight tunnel connecting the two cities would have a maximum depth of 137 km, so we would need to dig a very deep tunnel to get the minimal transit time.

The hypocycloid. A hypocycloid is the curve generated by a point on a circle which is moving inside a second circle without slipping. Let the outer circle have a radius a and the
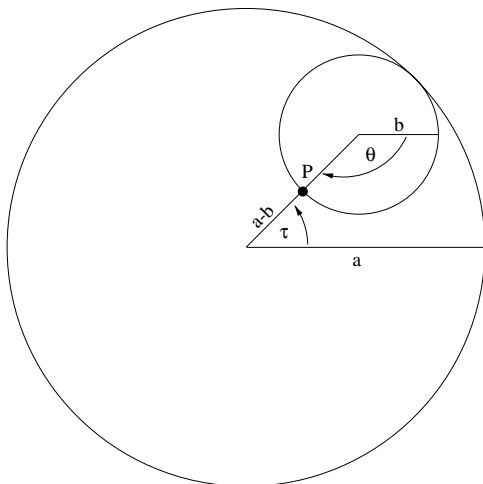
inner circle have a radius b. After the inner circle has rotated through an angle θ, the center of the inner circle has rotated through an angle τ. If the inner circle rolls without slipping, then $b\theta = (a - b)\tau$. The coordinates of a point P on the inner circle are then

$$x = (a - b)\cos\tau + b\cos\theta = (a - b)\cos\tau + b\cos\left[\left(\frac{a-b}{b}\right)\tau\right], \qquad (27)$$

$$y = (a - b)\sin\tau - b\sin\theta = (a - b)\sin\tau - b\sin\left[\left(\frac{a-b}{b}\right)\tau\right]. \qquad (28)$$

The plot below is a hypocycloid with $a = 3$ and $b = 1$ (the "deltoid").